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ON BUTLER'S UNIMODALITY RESULT

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For a partition λ, let $\alpha_\lambda(i;p)$ denote the number of subgroups of order p^i in a finite abelian p-group of type λ. Then $\alpha_\lambda(i;p)$ is a polynomial in p with nonnegative coefficients, which depends only on λ and i. Butler proved that $\alpha_\lambda(i;p)-\alpha_\lambda(i-1;p)$ where $1\le i\le|\lambda|/2$ has nonnegative coefficients. We prove this fact by using formulas shown by Stehling.

A partition is any sequence $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r,\dots)$ of nonnegative integers containing only finitely many nonzero terms that satisfy

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge \cdots.$$

If $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r,\dots)$ is a partition, $|\lambda|$ denotes the sum of the nonzero λ_i:

$$|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_r+\cdots,$$

and λ is called a partition of $|\lambda|$. For each partition $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r,0,\dots)$, we denote by $\alpha_\lambda(i;p)$, where p is a prime integer, the number of subgroups of order p^i in the direct product of cyclic p-groups

$$\mathbf{Z}/p^{\lambda_1}\mathbf{Z}\times\mathbf{Z}/p^{\lambda_2}\mathbf{Z}\times\cdots\times\mathbf{Z}/p^{\lambda_r}\mathbf{Z}$$

that is called a finite abelian p-group of type λ. Then $\alpha_\lambda(i;p)$ is a polynomial in p with nonnegative coefficients, which depends only on λ and i ([1], [2]). Let $\alpha_\lambda(i;p)=0$ if either $i<0$ or $i>|\lambda|$. It is well known that

$$\alpha_\lambda(i;p)=\alpha_\lambda(|\lambda|-i;p).$$

The purpose of this paper is to give a proof of the theorem below by using formulas shown in [2].

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Theorem 1. ([1]) *Let λ be a partition of s, and let i be an integer such that $1 \le i \le s/2$. Then $\alpha_\lambda(i;p) - \alpha_\lambda(i-1;p)$ has nonnegative coefficients.*

For each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ such that $|\lambda| > 0$, let

$$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \ldots)$$

where k is the largest number satisfying $\lambda_k = \lambda_1$, and let

$$\hat{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_r, \ldots).$$

When $\lambda = (0, 0, \ldots)$, let $\hat{\lambda} = \lambda$. For each partition λ of a positive integer s, by [2, Theorem 1],

$$\alpha_\lambda(i;p) = \alpha_{\tilde{\lambda}}(i;p) + p^{s-i}\alpha_{\hat{\lambda}}(s-i;p),$$

which is equivalent to the following.

Lemma 1. ([2, Corollary]) *Let λ be a partition such that $|\lambda| > 0$. Then*

$$\alpha_\lambda(i;p) = \alpha_{\tilde{\lambda}}(i-1;p) + p^i\alpha_{\hat{\lambda}}(i;p).$$

Combining these results, we have the following.

Lemma 2. *Let λ be a partition of a nonnegative integer s. Then*

$$\alpha_\lambda(i;p) - \alpha_\lambda(i-1;p) = p^i\alpha_{\hat{\lambda}}(i;p) - p^{s-i+1}\alpha_{\hat{\lambda}}(s-i+1;p).$$

We provide the following.

Lemma 3. *Let λ be a partition, and let $|\hat{\lambda}| = t$. Suppose that Theorem 1 holds for any partition μ satisfying $|\mu| \le t$. Let i be an integer, and let k be a nonnegative integer such that $2i - t \le k$. Then $\alpha_\lambda(i;p) - p^k\alpha_\lambda(i-k;p)$ has nonnegative coefficients.*

Proof. Let $|\lambda| = s$, and we use induction on s. The lemma is true if $s = 0$. Suppose that $s \ge 1$. Then it follows from Lemma 1 that

$$\alpha_\lambda(i;p) - p^k\alpha_\lambda(i-k;p) = \alpha_{\tilde{\lambda}}(i-1;p) - p^k\alpha_{\tilde{\lambda}}(i-1-k;p) + p^i\Big\{\alpha_{\hat{\lambda}}(i;p) - \alpha_{\hat{\lambda}}(i-k;p)\Big\}.$$

Since $k \ge 2i - t > 2(i-1) - (t-1)$, it follows from the inductive assumption that $\alpha_{\tilde{\lambda}}(i-1;p) - p^k\alpha_{\tilde{\lambda}}(i-1-k;p)$ has nonnegative coefficients. If $i \le t/2$, the hypothesis yields that $\alpha_{\hat{\lambda}}(i;p) - \alpha_{\hat{\lambda}}(i-k;p)$ has nonnegative coefficients. Furthermore, if $i > t/2$, $\alpha_{\hat{\lambda}}(t-i;p) - \alpha_{\hat{\lambda}}(i-k;p)$ has nonnegative coefficients, because $i-k \le t-i < t/2$. Since $\alpha_{\hat{\lambda}}(i;p) = \alpha_{\hat{\lambda}}(t-i;p)$, it follows that $\alpha_{\hat{\lambda}}(i;p) - \alpha_{\hat{\lambda}}(i-k;p)$ has nonnegative coefficients in any case. The result follows from these facts, thereby completing the proof of Lemma 3. ∎

Using the lemmas above, let us prove Theorem 1.

Proof of Theorem 1. We use induction on s. The result is clear if $s=0$. Suppose that $s\geq 1$. Let $|\hat{\lambda}|=t$. Then Lemma 2 yields that

$$\alpha_\lambda(i;p)-\alpha_\lambda(i-1;p)=p^i\left\{\alpha_{\hat{\lambda}}(t-i;p)-p^{s-2i+1}\alpha_{\hat{\lambda}}(t-s+i-1;p)\right\}.$$

Here, $t-s+i-1=(t-i)-(s-2i+1)$. Since $i\leq s/2$, it follows that $s-2i+1>0$. Furthermore, we obtain $s-2i+1>2(t-i)-(2t-s)$. Then it follows from the inductive assumption and Lemma 3 that $\alpha_{\hat{\lambda}}(t-i;p)-p^{s-2i+1}\alpha_{\hat{\lambda}}(t-s+i-1;p)$ has nonnegative coefficients. We have thus completed the proof of Theorem 1. ∎

The next theorem follows from Theorem 1 and Lemma 3, which is equivalent to Theorem 1.

Theorem 2. *Let λ be a partition, and let $|\hat{\lambda}|=t$. Let i be an integer, and let k be a nonnegative integer such that $2i-t\leq k$. Then $\alpha_\lambda(i;p)-p^k\alpha_\lambda(i-k;p)$ has nonnegative coefficients.*

References

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